Deriving Escape Velocity

We can derive escape velocity from Newton’s gravity force law:

\[ F = -G \cdot \frac{m_1 \cdot m_2}{r^2} \]

If we replace force \( F \) with the classic definition of Newton’s second law \( m \cdot a \), then we get:

\[ m_1 \cdot a = -G \cdot \frac{m_1 \cdot m_2}{r^2} \]

Cancelling terms, we have the general equation for the radial (centripetal) acceleration of a single, point mass (here we replace \( m_2 \) with \( m \)):

\[ a = -G \cdot \frac{m}{r^2} \]

Escape velocity is the velocity that lets us leave the surface of a mass and never return. This means that we always have a positive radial velocity and that radial velocity only approaches zero as distance from the mass approaches infinity. We obtain escape velocity by integrating this equation with respect to \( r \) from \( r = r_{\text{surface}} \) to \( r = \infty \):

\[ \int_{r=r_{\text{surface}}}^{r=\infty} a \cdot dr = \int_{r=r_{\text{surface}}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr \]

The expression on the right-hand side is straightforward; however, the expression on the left-hand side requires some adjustment. We start by replacing acceleration \( a \) with its definition: \( \frac{dv}{dt} \)

\[ \int_{r=r_{\text{surface}}}^{r=\infty} \frac{dv}{dt} \cdot dr = \int_{r=r_{\text{surface}}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr \]

Rearranging we get:

\[ \int_{r=r_{\text{surface}}}^{r=\infty} \frac{dr}{dt} \cdot dv = \int_{r=r_{\text{surface}}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr \]

The derivative \( \frac{dr}{dt} \) is simply velocity \( v \). This now gives us:

\[ \int_{r=r_{\text{surface}}}^{r=\infty} v \cdot dv = \int_{r=r_{\text{surface}}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr \]
To complete the adjustment, we must alter the limits of integration for the change of variable. At
\( r = r_{\text{surface}} \) we have \( v = v_{\text{escape}} \); and for \( r = \infty \), we have \( v = 0 \), the definition of an escape velocity at
infinity. This final change gives us:

\[
\int_{v = v_{\text{escape}}}^{v = 0} v \cdot dv = \int_{r = r_{\text{surface}}}^{r = \infty} -G \cdot \frac{m}{r^2} \cdot dr
\]

Which we now integrate:

\[
\frac{1}{2} \cdot v^2 \bigg|_{v = v_{\text{escape}}}^{v = 0} = G \cdot \frac{m}{r} \bigg|_{r = \infty}^{r = r_{\text{surface}}}
\]

And evaluate:

\[
0 - \frac{1}{2} \left( v_{\text{escape}} \right)^2 = 0 - G \cdot \frac{m}{r_{\text{surface}}}
\]

And solve for \( v_{\text{escape}} \):

\[
v_{\text{escape}} = \sqrt{\frac{2 \cdot G \cdot m}{r_{\text{surface}}}}
\]

This is the standard equation for escape velocity.