Deriving Escape Velocity

We can derive escape velocity from Newton's gravity force law:

$$F = -G \cdot \frac{m_1 \cdot m_2}{r^2}$$

If we replace force F with the classic definition of Newton's second law $m \cdot a$, then we get:

$$m_1 \cdot a = -G \cdot \frac{m_1 \cdot m_2}{r^2}$$

Cancelling terms, we have the general equation for the radial (centripetal) acceleration of a single, point mass (here we replace m_2 with m):

$$a = -G \cdot \frac{m}{r^2}$$

Escape velocity is the velocity that lets us leave the surface of a mass and never return. This means that we always have a positive radial velocity and that radial velocity only approaches zero as distance from the mass approaches infinity. We obtain escape velocity by integrating this equation with respect to r from $r = r_{surface}$ to $r = \infty$:

$$\int_{r=r_{surface}}^{r=\infty} a \cdot dr = \int_{r=r_{surface}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr$$

The expression on the right-hand side is straightforward; however, the expression on the left-hand side requires some adjustment. We start by replacing acceleration a with its definition: $\frac{dv}{dt}$

$$\int_{r=r_{surface}}^{r=\infty} \frac{dv}{dt} \cdot dr = \int_{r=r_{surface}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr$$

Rearranging we get:

$$\int_{r=r_{surface}}^{r=\infty} \frac{dr}{dt} \cdot dv = \int_{r=r_{surface}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr$$

The derivative $\frac{dr}{dt}$ is simply velocity v. This now gives us:

$$\int_{r=r_{surface}}^{r=\infty} v \cdot dv = \int_{r=r_{surface}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr$$

To complete the adjustment, we must alter the limits of integration for the change of variable. At $r = r_{surface}$ we have $v = v_{escape}$; and for $r = \infty$, we have $v = 0$, the definition of an escape velocity at infinity. This final change gives us:

$$\int_{v=v_{escape}}^{v=0} v \cdot dv = \int_{r=r_{surface}}^{r=\infty} -G \cdot \frac{m}{r^2} \cdot dr$$

Which we now integrate:

$$\tfrac{1}{2} \cdot v^2 \Big|_{v=v_{escape}}^{v=0} = G \cdot \frac{m}{r} \Big|_{r=r_{surface}}^{r=\infty}$$

And evaluate:

$$0 - \tfrac{1}{2} \cdot \left(v_{escape}\right)^2 = 0 - G \cdot \frac{m}{r_{surface}}$$

And solve for v_{escape}:

$$v_{escape} = \sqrt{\frac{2 \cdot G \cdot m}{r_{surface}}}$$

This is the standard equation for escape velocity.